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Jan 51

NACA TN 2253

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2253

ON A SOURCE-SINK METHOD FOR THE SOLUTION OF THE PRANDTL-BUSEMANN ITERATION EQUATIONS IN TWO-DIMENSIONAL COMPRESSIBLE FLOW

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Washington
December 1950

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ON A SOURCE-SINK METHOD FOR THE SOLUTION OF THE PRANDTL-BUSEMANN ITERATION EQUATIONS IN TWO-DIMENSIONAL COMPRESSIBLE FLOW

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SUMMARY

The recently derived particular integrals of the Prandtl-Busemann iteration equations make possible the extension of the familiar source-sink concept to the solution of the higher-order iteration equations for the subsonic potential flow over thin sharp-nose symmetric two-dimensional profiles. An explicit expression is derived for the second-order velocity potential and velocity components and a method for obtaining the higher-order terms is indicated. The velocity at the surface of the Kaplan bump is evaluated to illustrate the method.

INTRODUCTION

Iteration methods have been used extensively during recent years for the calculation of compressible flows. The Prandtl-Busemann or Ackeret iteration process is the one most often applied in aeronautics and is the one considered herein. In the past this iteration technique has been used primarily to calculate the flow past specific profiles, each profile presenting a distinct problem involving a great amount of labor in its solution.

Van Dyke (reference 1) has given the particular integral for the second-order Prandtl-Busemann iteration equation and recently Kaplan (reference 2) has obtained the particular integrals for both the second-order and third-order equations and has described a method for obtaining higher-order ones as well. With the use of these particular integrals, the higher-order solutions for the flow over sharp-nose symmetric bodies in two-dimensional flow are easily obtained by the method to be described herein, and the calculation of the surface velocity or pressure distribution for a specific profile requires only an integration. The procedure to be described involves considerably less labor than formerly required and also has the advantage of simplicity of treatment since the familiar concept of a source-sink distribution in the plane of symmetry is employed in the analysis.

A general expression for the second-order velocity potential for symmetric profiles is developed and, as an example, the flow over the Kaplan bump is treated in detail. The procedure may be extended to obtain the third- and higher-order perturbation velocities.

ANALYSIS

The Prandtl-Busemann iteration equations for obtaining approximate solutions to the exact nonlinear equation governing the two-dimensional flow of a compressible fluid are briefly developed as follows (reference 2):

The differential equation for the velocity potential Φ is

$$\left(1 - \frac{u^2}{a^2}\right)\Phi_{xx} - 2\,\frac{uv}{a^2}\,\Phi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\Phi_{yy} = 0 \tag{1}$$

where

x,y rectangular Cartesian coordinates in flow plane

u,v velocity components along x- and y-axes, respectively

a local speed of sound

and the subscripts x and y denote partial differentiation with respect to these designated variables. With the introduction of a characteristic length $c/2$, where c is the chord, and the undisturbed stream velocity U as the unit of velocity, the quantities x, y, u, v, and Φ are used hereinafter to denote, respectively, the nondimensional quantities $2x/c$, $2y/c$, u/U, v/U, and $2\Phi/Uc$ while a retains its original meaning.

Let

$$Y = tY_1(x) + t^2Y_2(x) + O(t^3) \qquad \left(-1 < x < 1\right)$$

and

$$Y = 0 \qquad (x \geqq 1;\ x \leqq -1)$$

where $O(t^3)$ denotes terms of the order t^3, define a slender symmetric body of thickness coefficient t lying on the x-axis between -1 and 1.

The appropriate boundary conditions for the flow over a two-dimensional body in an unbounded stream are:

At infinity

$$\Phi_x = 1$$

$$\Phi_y = 0$$

and on the body

$$\Phi_y(x,Y) = \Phi_x(x,Y)\frac{dY}{dx} \tag{2}$$

It is now assumed that Φ may be expanded in the form

$$\Phi = x + \phi_1 + \phi_2 + \ldots$$

where ϕ_{n+1} and its derivatives are small compared with ϕ_n, and ϕ_n is of the order t^n. Then, when the expansion for Φ is substituted in the differential equation (1), the linear equations for ϕ_1 and ϕ_2 are

$$\beta^2\phi_{1xx} + \phi_{1yy} = 0 \tag{3}$$

$$\beta^2\phi_{2xx} + \phi_{2yy} = 2M_\infty^2\left[(1 + \sigma)\beta^2\phi_{1xx}\phi_{1x} + \phi_{1xy}\phi_{1y}\right] \tag{4}$$

where $\sigma = \frac{\gamma + 1}{2}\frac{M_\infty^2}{\beta^2}$, $\beta^2 = 1 - M_\infty^2$, M_∞ is the Mach number at infinity, and γ is the ratio of specific heats at constant pressure and constant volume.

The boundary conditions for ϕ_1 and ϕ_2, to the order t and t^2, respectively, are:

On the body

$$\phi_{1y}(x,0) = t\frac{dY_1}{dx} \tag{5}$$

$$\phi_{2y}(x,0) = t^2\frac{dY_2}{dx} + t\frac{dY_1}{dx}\phi_{1x}(x,0) + \beta^2tY_1\phi_{1xx}(x,0) \tag{6}$$

and at infinity

$$\phi_{1x} = \phi_{1y} = \phi_{2x} = \phi_{2y} = 0 \tag{7}$$

The well-known source-sink solution of the differential equation for ϕ_1, valid for symmetric profiles and satisfying the boundary conditions given by equations (5) and (7), is (see reference 3, for example)

$$\phi_1(x,y) = \frac{t}{\beta\pi}\int_{-1}^{1} \frac{dY_1}{d\xi} \log_e\sqrt{(x - \xi)^2 + \beta^2y^2}\, d\xi \tag{8}$$

The solution of the nonhomogeneous differential equation for ϕ_2, equation (4), can be expressed as the sum of the particular integral $\psi_2(x,y)$ and a function $\varphi_2(x,y)$ satisfying the homogeneous equation.

$$\beta^2\varphi_{2xx} + \varphi_{2yy} = 0$$

The particular integral ψ_2 of equation (4) (reference 2) is

$$\psi_2(x,y) = M_\infty^2\phi_{1x}\left[\left(1 + \frac{\sigma}{2}\right)\phi_1 - \frac{\sigma}{2}\, y\phi_{1y}\right] \tag{9}$$

With the use of equations (8) and (9), ψ_{2x} and ψ_{2y} are easily shown to vanish at infinity and the boundary conditions for φ_2, from equations (6) and (7), become:

On the body

$$\varphi_{2x}(x,0) = \frac{dF(x)}{dx}$$

where

$$\frac{dF(x)}{dx} \equiv \beta^2t\left[Y_1\phi_{1xx}(x,0) + \frac{dY_1}{dx}\,\phi_{1x}(x,0)\right] + t^2\,\frac{dY_2}{dx} - M_\infty^2\left(1 + \frac{\sigma}{2}\right)\phi_{1xy}(x,0)\phi_1(x,0)$$

and at infinity

$$\varphi_{2x} = \varphi_{2y} = 0$$

It is to be understood that all differentiations are made prior to setting y equal to zero.

The differential equation and boundary conditions for φ_2 are of the same form as for ϕ_1. Therefore, the solution for ϕ_2 may be expressed as

$$\phi_2(x,y) = \frac{1}{\beta\pi}\int_{-1}^{1}\frac{dF(\xi)}{d\xi}\log_e\sqrt{(x - \xi)^2 + \beta^2y^2}\,d\xi + \psi_2(x,y)$$

provided that $\int_{-1}^{1}\left|F(\xi)\right|d\xi$ converges and $F(\xi)$ is sectionally continuous. Since the source-sink solution may be obtained by using the concepts of Fourier integrals, it follows that the function $F(\xi)$ must satisfy the conditions which are required for a function to be represented in a Fourier integral. These conditions are sufficient to insure the existence of the solution although somewhat broader conditions for $F(\xi)$ are known. These conditions seem to exclude the representation of the second-order flow past blunt bodies by a continuous distribution of sources and sinks on the axis alone.

Then the velocity potential to the order t^2 is

$$\Phi(x,y) = x + \frac{1}{\beta\pi}\int_{-1}^{1}\left[t\frac{dY_1}{d\xi} + \frac{dF(\xi)}{d\xi}\right]\log_e\sqrt{(x - \xi)^2 + \beta^2y^2}\,d\xi + \psi_2(x,y) + O(t^3) \tag{10}$$

and the velocity components on the body to the same order are

$$\begin{aligned} u &= \Phi_x(x,Y) \\ &= 1 + \frac{1}{\beta\pi}P\int_{-1}^{1}\left[t\frac{dY_1}{d\xi} + \frac{dF(\xi)}{d\xi}\right]\frac{d\xi}{x - \xi} + tY_1\phi_{1xy}(x,0) + \\ &\quad M_\infty^2\left(1 + \frac{\sigma}{2}\right)\left\{\phi_{1xx}(x,0)\phi_1(x,0) + \left[\phi_{1x}(x,0)\right]^2\right\} + O(t^3) \end{aligned} \tag{11}$$

$$\begin{aligned} v &= \Phi_y(x,Y) \\ &= t\frac{dY_1}{dx}\left[1 + \phi_{1x}(x,0)\right] + t^2\frac{dY_2}{dx} + O(t^3) \end{aligned} \tag{12}$$

where P denotes the Cauchy principal value of the integral.

Once ϕ_1 is determined for a given profile, the second-order velocity potential and the corresponding velocity components can be found from equations (10), (11), and (12).

The third- and higher-order terms for the velocity potential or velocity components can be determined in the same manner as the second-order terms. The third-order term for the velocity potential is written as the sum of a function satisfying the homogeneous differential equation and a particular integral (from reference 2). Then the boundary conditions for the homogeneous equation are again in a form similar to those for ϕ_1 and the third-order solution is the sum of the particular integral and an expression for a source distribution with the source strength determined by the boundary conditions.

It appears that this method, as developed herein, is not applicable for evaluating the flow over all bodies. For blunt bodies, the method must be refined to include other solutions in conjunction with the source-sink solution in order to satisfy the boundary conditions. It might be noted that the troublesome terms in $F(\xi)$ arise from the particular integral. Since the particular integral is multiplied by M^2, the difficulty encountered is a Mach number effect.

It does not seem possible to obtain a second-order solution for the flow past a sharp-nose body at a small angle of attack (to the second order in thickness and first order in angle of attack) by only a source-sink and vortex distribution on the axis. An integral equation must be solved to obtain the function in the lift problem corresponding to $F(\xi)$ in the thickness problem. Therefore, it is difficult to make any general statements about the conditions this function must satisfy. However, the particular integral seems to introduce a singularity in the expression for the lift coefficient which cannot be canceled by the solution to the homogeneous equation. Here again the method must be refined in order to obtain the second-order lift solutions.

In general, the integral for ϕ_2 or its derivatives are more difficult to evaluate than those for ϕ_1 and in some cases it may be desirable to approximate the integral for ϕ_2. When ϕ_1 is known, the integrand of the integral for ϕ_2 or its derivatives may be expanded in a Taylor series and the integration performed term by term.

The Taylor expansion of the function $\frac{dF(\xi)}{d\xi}$ about the point x is

$$\frac{dF(\xi)}{d\xi} = \sum_{n=0}^{\infty} \frac{(\xi - x)^n}{n!} F^{(n+1)}(x) \tag{13}$$

where

$$\frac{dF(x)}{dx} = t^2 \frac{dY_2}{dx} + \beta^2 tY_1 \phi_{1xx}(x,0) + \beta^2 t \frac{dY_1}{dx} \phi_{1x}(x,0) -$$

$$M_\infty^2\left(1 + \frac{\sigma}{2}\right)\phi_{1xy}(x,0)\phi_1(x,0)$$

$$F^{(n)}(x) = \frac{d^n}{dx^n} F(x)$$

Then, when $\frac{dF(\xi)}{d\xi}$ of equation (13) is substituted in equation (11) and the resulting equation is integrated term by term, the u component of velocity becomes

$$u = 1 + \phi_{1x}(x,0) + t^2 Y_1 \frac{dY_1}{dx} + M_\infty^2\left(1 + \frac{\sigma}{2}\right)\left[\phi_{1x}^2(x,0) + \phi_{1xx}(x,0)\phi_1(x,0)\right] -$$

$$\frac{1}{\beta\pi}\left[\frac{dF(x)}{dx} \log_e \frac{1 - x}{1 + x} + \sum_{n=1}^{\infty} \frac{F^{(n+1)}(x)}{n(n!)}\left[(1 - x)^n - (-1 - x)^n\right]\right]$$

$$(-1 < x < 1)$$

CALCULATION OF THE FLOW PAST THE KAPLAN BUMP

As an example of the use of the second-order equations, the velocity at the surface of the Kaplan bump is evaluated. The equation of the bump is

$$Y(x) = t(1 - x^2)^{3/2} - 3x^2t^2(1 - x^2)^{3/2} + O(t^3)$$

$$Y_1 = (1 - x^2)^{3/2}$$

$$Y_2 = -3x^2(1 - x^2)^{3/2}$$

From equation (8), the expression for $\phi_{1x}(x,0)$ is

$$\phi_{1x}(x,0) = \frac{1}{\beta\pi} P \int_{-1}^{1} \frac{dY_1}{d\xi} \frac{d\xi}{x - \xi} = -\frac{3t}{\beta\pi} P \int_{-1}^{1} \frac{\xi\sqrt{1 - \xi^2}}{x - \xi} d\xi$$

By the substitution of $\xi = \cos\theta$, the expression for $\phi_{1x}(x,0)$ becomes

$$\phi_{1x}(x,0) = \frac{3t}{\beta\pi}\left(\int_0^{\pi} \sin^2\theta \, d\theta - xP \int_0^{\pi} \frac{\sin^2\theta \, d\theta}{x - \cos\theta}\right)$$

With $\sin^2\theta$ replaced by $\frac{1}{2}(1 - \cos 2\theta)$ and with the use of the following integral

$$P \int_0^{\pi} \frac{\cos n\theta \, d\theta}{\cos\alpha - \cos\theta} = \frac{-\pi \sin n\alpha}{\sin\alpha} \tag{14}$$

the expressions for $\phi_{1x}(x,0)$ and $\phi_1(x,0)$ are

$$\phi_{1x}(x,0) = \frac{3t}{\beta}\left(\frac{1}{2} - x^2\right) \qquad (-1 \leqq x \leqq 1)$$

$$\phi_1(x,0) = \frac{t}{\beta}\left(\frac{3}{2} x - x^3\right) \qquad (-1 \leqq x \leqq 1)$$

The u component of velocity (equation (11)) is

$$\begin{aligned} u &= \Phi_x(x,Y) \\ &= 1 + \frac{1}{\beta\pi} P \int_{-1}^{1}\left[t \frac{dY_1}{d\xi} + \frac{dF(\xi)}{d\xi}\right]\frac{d\xi}{x - \xi} + tY_1\phi_{1xy}(x,0) + \\ &\quad M_\infty^2\left(1 + \frac{\sigma}{2}\right)\left\{\phi_{1xx}(x,0)\phi_1(x,0) + \left[\phi_{1x}(x,0)\right]^2\right\} + O(t^3) \end{aligned} \tag{15}$$

with

$$\frac{dF(x)}{dx} = \beta^2 t\left[Y_1\phi_{1xx}(x,0) + \frac{dY_1}{dx}\phi_{1x}(x,0)\right] + t^2 \frac{dY_2}{dx} - M_\infty^2\left(1 + \frac{\sigma}{2}\right)\phi_{1xy}(x,0)\phi_1(x,0)$$

where $\phi_{1xy}(x,0) = t\,\frac{d^2Y_1(x)}{dx^2}$. The various terms arising in $\frac{dF(x)}{dx}$ are

$$\left.\begin{aligned}
Y_1\phi_{1xx}(x,0) &= -\frac{6tx}{\beta}(1 - x^2)^{3/2} = \frac{6t}{\beta}(x^3 - x)\sqrt{1 - x^2} \\
\frac{dY_1}{dx}\,\phi_{1x}(x,0) &= -\frac{9t}{\beta}\left(\frac{x}{2} - x^3\right)\sqrt{1 - x^2} \\
\frac{dY_2}{dx} &= (-6x + 15x^3)\sqrt{1 - x^2} \\
\frac{d^2Y_1}{dx^2}\,\phi_1(x,0) &= \frac{-3t\left(\frac{3}{2}x - 4x^3 + 2x^5\right)}{\sqrt{1 - x^2}}
\end{aligned}\right\} \tag{16}$$

The following integrals will be required:

$$\left.\begin{aligned}
P\int_{-1}^{1}\xi\,\frac{\sqrt{1 - \xi^2}\,d\xi}{x - \xi} &= P\int_{0}^{\pi}\frac{\cos\theta\,\sin^2\theta\,d\theta}{x - \cos\theta} = \pi\left(-\frac{1}{2} + x^2\right) \\
P\int_{-1}^{1}\xi^3\,\frac{\sqrt{1 - \xi^2}\,d\xi}{x - \xi} &= P\int_{0}^{\pi}\frac{\cos^3\theta\,\sin^2\theta\,d\theta}{x - \cos\theta} = \pi\left(-\frac{1}{8} - \frac{x^2}{2} + x^4\right) \\
P\int_{-1}^{1}\frac{\xi\,d\xi}{\sqrt{1 - \xi^2}(x - \xi)} &= P\int_{0}^{\pi}\frac{\cos\theta\,d\theta}{x - \cos\theta} = -\pi \\
P\int_{-1}^{1}\frac{\xi^3 d\xi}{\sqrt{1 - \xi^2}(x - \xi)} &= P\int_{0}^{\pi}\frac{\cos^3\theta\,d\theta}{x - \cos\theta} = -\pi\left(\frac{1}{2} + x^2\right) \\
P\int_{-1}^{1}\frac{\xi^5 d\xi}{\sqrt{1 - \xi^2}(x - \xi)} &= P\int_{0}^{\pi}\frac{\cos^5\theta\,d\theta}{x - \cos\theta} = -\pi\left(\frac{3}{8} + \frac{x^2}{2} + x^4\right)
\end{aligned}\right\} \tag{17}$$

where repeated use has been made of equation (14).

From equation (15), with the use of equations (16) and (17), and with

$$Y_1\,\frac{d^2Y_1}{dx^2} = -3t^2\left(1 - 3x^2 + 2x^4\right)$$

$$\left[\phi_{1x}(x,0)\right]^2 + \phi_1(x,0)\phi_{1xx}(x,0) = \frac{3t^2}{\beta^2}\left(\frac{3}{4} - 6x^2 + 5x^4\right)$$

the x-component of velocity on the body, to the order t^2, is

$$u(x,Y) = 1 - \frac{3}{2}\frac{t}{\beta}\left(2x^2 - 1\right) + \frac{3t^2}{\beta^2}\left[3x^4 - 3x^2 + \frac{1}{2} + 5\beta x^4 - \frac{9}{2}\beta x^2 + \frac{3}{8}\beta - \frac{3}{8}\beta^2 + M_\infty^2\sigma\left(\frac{3}{2}x^4 - \frac{3}{2}x^2 + \frac{1}{4}\right)\right]$$

in agreement with reference 4.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., October 23, 1950

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NACA-Langley - 12-29-50 - 1025

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NACA TN 2253
December 1950

(Abstract on Reverse Side)

Abstract

The recently derived particular integrals of the Prandtl-Busemann iteration equations make possible the extension of the familiar source-sink concept to the solution of the higher-order iteration equations for the subsonic potential flow over thin symmetric two-dimensional profiles. An explicit expression is derived for the second-order velocity potential and velocity components and a method for obtaining the higher-order terms is indicated. The velocity at the surface of the Kaplan bump is evaluated to illustrate the method.

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